NOTES ON SPLINE FUNCTIONS III: ON THE CONVERGENCE OF THE INTERPOLATING CARDINAL SPLINES AS THEIR DEGREE TENDS TO INFINITY†

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ABSTRACT

It is shown that for entire functions $f(x)$ defined by a Fourier-Stieltjes integral (9) the cardinal spline $S_m(x)$ of the odd degree $2m-1$, which interpolates $f(x)$ at all integers, converges to $f(x)$ as m tends to infinity. Properties of the exponential Euler spline are used in the proof.

1. Introduction

Let $n = 2m - 1$ be an odd integer and let $\mathcal{S}_n = \{S(x)\}$ denote the class of spline functions $S(x)$ of degree $n = 2m - 1$, with knots at the integers and of the continuity class $C^{2m-2}(\mathbb{R})$. Within this class we wish to interpolate a prescribed bi-infinite sequence (y_ν) of numbers for $-\infty < \nu < \infty$, that is, $S(\nu) = y_\nu$ for all integers ν. We know (see, for example, [6, Lec. 4]) that if y_ν grows at most like a power of $|\nu|$ as $|\nu| \to \infty$, then there is a unique $S_m(x) \in \mathcal{S}_{2m-1}$ such that $S_m(x)$ grows at most like a power of $|x|$ as $|x| \to \infty$ which satisfies

$$S_m(\nu) = y_\nu \text{ for all } \nu.$$

We are interested in cases when $S_m(x)$ converges to a limit function as m approaches infinity. Two such cases are presently known.

(i) The sequence (y_ν) is periodic with period k.

(ii) $(y_\nu) \in l_2$, hence $\sum |y_\nu|^2 < \infty$. For case (i) refer to [3], [1], [4], and for case (ii) to [5] and [6, Lec. 9]. Since, for the purposes of this paper, we are concerned

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exclusively with cardinal splines, we shall specialize the results of [1] and [4] to the case of equidistant knots. Let us now state the relevant results in detail.

For Case (i), the sequence (y_ν) is periodic with period k, hence $y_\nu = y_{\nu+k}$ for all ν. The results depend on the parity of k.

For Case (i), $k = 2l + 1$ is odd. It is well known that the periodic sequence (y_ν) can be uniquely interpolated at the integers by a trigonometric polynomial of the form

$$T(x) = \sum_{\nu=-l}^{l} A_\nu \exp\left(\frac{2\pi i\nu x}{2l+1}\right) \tag{1}$$

with period $k = 2l + 1$. Specializing results from [1] and [4] to equidistant knots we obtain Theorem 1.

THEOREM 1. *As m approaches infinity*

$$\lim S_m(x) = T(x)$$

uniformly for all real x.

For Case (i), $k = 2l$ is even. In this case we know that (y_ν) can be interpolated by a unique trigonometric polynomial

$$T^*(x) = \sum_{\nu=-l}^{l} A_\nu \exp\left(\frac{2\pi i\nu x}{2l}\right) \tag{2}$$

such that

$$A_{-l} = A_l. \tag{3}$$

This is the proximal interpolant of [4]. Specializing results from [1] and [4] we obtain Theorem 2.

THEOREM 2. (*Quade and Collatz* [3]). *As m approaches infinity*

$$\lim S_m(x) = T^*(x)$$

uniformly for all real x.

For Case (ii), $(y_\nu) \in l_2$, or $\sum |y_\nu|^2 < \infty$. By the Riesz-Fischer theorem we can write

$$y_\nu = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\nu u} g(u)du, \text{ where } g(u) \in L_2(-\pi, \pi).$$

Specializing results from [5, Th. 2], or [6, Lec. 9] we obtain Theorem 3.

THEOREM 3. *If we define the entire function $f(x)$ by*

$$f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ixu} g(u) du \tag{4}$$

then

$$\lim_{m \to \infty} S_m(x) = f(x)$$

uniformly for all real x.

Reporting about these results in [6, Lec. 9] I conclude with the remark: "...The question arises as to the existence of a comprehensive theory that would cover these separate cases...". A comprehensive general discussion, as given in this paper, has proven much simpler than the proofs of either of the individual cases described above. This is so because of the properties of the so-called exponential Euler splines whose definition and essential properties are given as follows. (The reader is referred to [6, Lec. 2, 3] for further details.)

Let

$$t = \tau e^{iu}, \text{ for } -\pi < u < \pi, \ \tau > 0, \tag{5}$$

be a fixed parameter, assuming for the moment that $t \neq 1$. We denote by $A_n(x; t)$ the so-called exponential Euler polynomials defined by the generating function

$$\frac{t-1}{t-e^z} e^{xz} = \sum_{0}^{\infty} \frac{A_n(x; t)}{n!} z^n.$$

$A_n(x; t)$ is a monic polynomial of degree n in x with the property that $A_n(0; t) \neq 0$ for all t subject to (5). This allows us to make the following definition.

DEFINITION 4. We define the exponential Euler spline $S_n(x; t)$ by setting

$$S_n(x; t) = \frac{A_n(x; t)}{A_n(0; t)} \text{ for } 0 \leqq x < 1.$$

We extend its definition to all real values of x by means of the functional equation

$$S_n(x + 1; t) = t\, S_n(x; t) \text{ for all real } x.$$

Finally we complete its definition for $t = 1$ by setting $S_n(x; 1) = 1$ for all real x. The essential properties of $S_n(x; t)$ are stated in Lemma 5.

LEMMA 5. *$S_n(x; t)$ has four essential properties.*

(i) *Whether n is even or odd, $S_n(x; t)$ is a piecewise polynomial function of the continuity class $C^{n-1}(\mathbb{R})$. Thus*

$$S_n(x; t) \in \mathcal{S}_n.$$

(ii) *$S_n(x;t)$ interpolates the exponential $t^x = \tau^x e^{iux}$ at all integers, hence*

$$S_n(\nu;t) = t^\nu \text{ for all integers } \nu.$$

(iii) *The relation*

$$\lim_{n\to\infty} S_n(x;t) = t^x \tag{6}$$

holds.

(iv) *If $n = 2m-1$ is odd, then $A_n(0;-1) \neq 0$. We may let $t = -1$, a value forbidden heretofore. In fact*

$$S_{2m-1}(x;-1) = \mathscr{E}_{2m-1}(x)$$

is the well-known Euler spline of odd degree. If we restrict t to the unit circle, we may assume that

$$t = e^{iu}, \quad -\pi \leqq u \leqq \pi.$$

Then (6) can be sharpened to the inequality

$$\left| e^{iux} - S_{2m-1}(x;e^{iu}) \right| \leqq C_m \left(\frac{|u|}{\pi} \right)^{2m} \text{ for all real } x. \tag{7}$$

Here C_m is a constant which is less than three for all m. Some best values of C_m are

$$C_1 = \tfrac{1}{8}\pi^2, \quad C_2 = 1, \ C_3 = 1.$$

We conjecture that the best value is $C_m = 1$ for $m > 3$.

The inequality (7), with C_m replaced by the value 4, was first established by Golomb [2] *for values of u which are rational multiples of π.* This was a necessary restriction, since Golomb's discussion was devoted only to periodic spline interpolation. A proof of (7) for the present cardinal case is actually much simpler.

As a very special case of Theorem 2 we obtain that

$$\lim_{m\to\infty} S_{2m-1}(x;-1) = \cos \pi x \text{ uniformly for real } x. \tag{8}$$

This fact will be used in Section 2.

2. The main result

The presentation and derivation of our main reasult can now be briefly stated after our elaborate introduction. Consider the Fourier-Stieltjes transform

$$f(x) = \int_{-\pi}^{\pi} e^{ixu} d\alpha(u), \tag{9}$$

where $\alpha(u)$ is of bounded variation in $[-\pi, \pi]$. Clearly $f(x)$ is an entire function of exponential type less than or equal to π. Moreover, if we normalize $\alpha(u)$ in order to satisfy $2\alpha(u) = \alpha(u+0) + \alpha(u-0)$ for $-\pi < u < \pi$, then $\alpha(u)$ is defined uniquely by (9) up to an additive constant. If we define

$$\alpha_0(u) = \begin{cases} \alpha(-\pi+0) & \text{if} \quad u = -\pi, \\ \alpha(u) & \text{if} \quad -\pi < u < \pi, \\ \alpha(\pi - 0) & \text{if} \quad u = \pi, \end{cases}$$

then $\alpha_0(u)$ is also continuous at the endpoints $u = \pm\pi$. Setting

$$A = \alpha(-\pi+0) - \alpha(-\pi), \quad B = \alpha(\pi) - \alpha(\pi - 0) \tag{10}$$

we may write

$$f(x) = \int_{-\pi}^{\pi} e^{ixu} d\alpha_0(u) + Ae^{-\pi i x} + Be^{\pi i x}$$

or

$$f(x) = \int_{-\pi}^{\pi} e^{ixu} d\alpha_0(u) + (A+B)\cos \pi x + i(B-A)\sin \pi x. \tag{11}$$

THEOREM 6. *If $S_m(x)$ denotes the unique bounded element of $\mathcal{S}_{2m-1}$ that interpolates $f(x)$ at all integer values of x, then*

$$\lim_{m\to\infty} S_m(x) = \int_{-\pi}^{\pi} e^{ixu} d\alpha_0(u) + (A+B)\cos \pi x$$

uniformly for all real x.

PROOF. We claim that the unique interpolating spline $S_m(x)$ which is bounded is also given by

$$S_m(x) = \int_{-\pi}^{\pi} S_{2m-1}(x; e^{iu}) \, d\alpha_0(u) + (A+B)\mathscr{E}_{2m-1}(x). \tag{12}$$

Indeed, observe first that $|S_{2m-1}(x; e^{iu})| \leqq 1$ for all real x, by [6, Lec. 3, Rel. (6.13)], then that the Riemann-Stieltjes sum

$$\Sigma S_{2m-1}(x; \exp(iv_j)(\alpha_0(u_j) - \alpha_0(u_{j-1})) \tag{13}$$

evidently is an element of $\mathcal{S}_{2m-1}$ that converges uniformly on refinement with respect to x in every finite interval of the x-axis. It follows that the $S_m(x)$, defined by (12), is an element of $\mathcal{S}_{2m-1}$. To show that it interpolates $f(x)$, for $x = v$ an integer, note that (12) implies that

$$S_m(v) = \int_{-\pi}^{\pi} e^{ivu} d\alpha_0(u) + (A+B)\cos \pi v + i(B-A)\sin \pi v = f(v)$$

by (11). This proves our claim in the opening statement of the proof of Theorem 6.

We now define

$$f_0(x) = \int_{-\pi}^{\pi} e^{ixu}\, d\alpha_0(u) + (A+B)\cos \pi x. \tag{14}$$

Subtracting (12) from (14) we obtain

$$f_0(x) - S_m(x) = \int_{-\pi}^{\pi} (e^{ixu} - S_{2m-1}(x; e^{iu}))d\alpha_0(u) + (A+B)(\cos \pi x - \mathscr{E}_{2m-1}(x)),$$

from which

$$\left|f_0(x) - S_m(x)\right| \leqq \int_{-\pi}^{\pi} \left|e^{iux} - S_{2m-1}(x; e^{iu})\right| \left|d\alpha_0(u)\right| + $$
$$+ \left|A+B\right| \left|\cos \pi x - \mathscr{E}_{2m-1}(x)\right|.$$

Using sup-norms on $\mathbb{R}$, (7) shows that

$$\left\|f_0(x) - S_m(x)\right\|_\infty \leqq 3\int_{-\pi}^{\pi} \left(\frac{|u|}{\pi}\right)^{2m} \left|d\alpha_0(u)\right| + \left|A+B\right| \cdot \left\|\cos \pi x - \mathscr{E}_{2m-1}(s)\right\|_\infty.$$

The last term on the right side approaches zero as m approaches infinity by (8). The integral approaches zero as well because of the continuity of $\alpha_0(u)$ at the endpoints $\pm\pi$. This also implies the continuity of the total variation of $\alpha_0(u)$ at those points which completes the proof of Theorem 6.

3. Special cases

In this section we wish to show how Theorem 6 furnishes the previously shown results as well as many new ones for special choices of the function $\alpha(u)$. We offer three such cases.

(i) (y_v) is a periodic sequence. Let $k = 2l+1$ be odd. The trigonometric polynomial (1) is evidently of the form (9) if $\alpha(u)$ is a suitable step-function. This step-function is continuous at $\pm\pi$ and Theorem 6 implies Theorem 1. If $k = 2l$ is even, then (2) also results from (9) for an appropriate step-function $\alpha(u)$. The relation (3) shows that its two jumps (10) are equal, hence $f(x) = f_0(x)$ and Theorem 6 implies Theorem 2.

(ii) $(y_v) \in l_2$. The function $f(x)$ as defined by (4), is also of the form required by Theorem 6. Since $g(u) \in L_2(-\pi, \pi)$ implies $g(u) \in L_1(-\pi, \pi)$, it suffices to set

$$\alpha(u) = \frac{1}{2\pi}\int_{-\pi}^{u} g(v)dv, \text{ for } -\pi \leqq u \leqq \pi.$$

Here $\alpha(u)$ is even absolutely continuous.

(iii) $F(x)$ is almost periodic in the sense of H. Bohr. If $\alpha(u)$, as defined in (9), is purely discontinuous, we obtain from (9) that with pairwise different λ_ν

$$f(x) = \sum_{\nu=1}^{\infty} A_\nu \exp(i\lambda_\nu x), \quad \sum_{1}^{\infty} |A_\nu| < \infty, \text{ for } -\pi \leqq \lambda_\nu \leqq \pi.$$

Of course the number of discontinuities may also be finite, but in any case $f(x)$ is almost periodic and has an absolutely-convergent Fourier series. In order to have the interpolating spline $S_m(x)$ converge to $f(x)$ we must assume that if $\lambda_1 = -\pi$, then (say) $\lambda_2 = \pi$, and $A_1 = A_2$, hence

$$f(x) = 2A_1 \cos \pi x + \sum_{3}^{\infty} A_\nu \exp(i\lambda_\nu x).$$

References

1. M. v. Golitschek, *On the convergence of interpolating periodic spline functions of high degree*, Numer. Math. **19** (1972), 146–154.
2. M. Golomb, *Approximation by periodic spline interpolants on uniform meshes*, J. Approximation Theory, **1** (1968), 26–65.
3. W. Quade and L. Collatz, *Zur Interpolationstheorie der reellen periodischen Funktionen*, Sitzunsgber. Preussischen Akad. Wiss., Phys.-Math. Kl. (1938), 383–429.
4. I. J. Schoenberg, *Notes on spline functions I. The limits of the interpolating periodic spline functions as their degree tends to infinity*, MRC T. S. R. ≠ 1219, April 1972 (to appear in the Indag. Math.).
5. I. J. Schoenberg, *Cardinal interpolation and spline functions VII. The behavior of cardinal spline interpolants as their degree tends to infinity*, MRC T. S.R. ≠ 1184, December 1971 (to appear in J. Analyse Math.).
6. I. J. Schoenberg, *Cardinal spline interpolation*, Regional Conference Series in Applied Mathematics, No. 12, 125 pages, S. I. A. M., Philadelphia, Pa., 1973.

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